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NONLINEAR ACOUSTICS

Hysteresis in an Acoustic Medium with Relaxing Nonlinearity and Viscosity

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Abstract—The loading and unloading waves propagating in a nonlinear relaxing and dissipative medium of the consolidated soil type are investigated. Solutions describing the waves in such a medium are constructed with the use of the Stokes method and the small-distance asymptotic approach. Explicit approximate solutions are obtained for different values of the relaxation and viscosity parameters. The influence of the type of the medium on the shape of the hysteretic curves is described.

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INTRODUCTION

Nonlinear wave processes, including those with hysteretic nonlinearity, had been much discussed in the literature. For example, one may note the papers [1, 2] and the review [3], as well as other publications cited below.

In connection with the topical problems of construction, the theory of nonlinear acoustic wave propagation in media with irreversible deformations, which are typical of consolidated soil, is of special importance. Such a theory should describe the propagation process by simultaneously taking into account the nonlinear relaxation and the viscosity of the medium. Solution of the problem of wave propagation with allowance for both of the aforementioned effects is rather difficult. The problem was analyzed in [4], but the analysis was not completed.

The present paper continues the study described in [4] and considers a simplified model, in which only the unloading wave is relaxing. For such a process, the following equation in terms of dimensionless variables was derived in [4]:

$$(P_z - PP_\theta)_\theta + \kappa[P_z - P_m(z)P_\theta] = 0. \qquad (1)$$

Here, P, z, and θ are the dimensionless pressure, distance, and time, respectively; $P_m(z)$ is the maximum dimensionless pressure; the subscripts z and θ denote differentiation with respect to the respective variables; and the dimensionless coefficient κ takes into account the relaxation. When $\kappa \ll 1$, the relaxation time is small, and precisely this case is considered below. Under this condition, Eq. (1) can be simplified:

$$P_z - PP_\infty = -\frac{1}{2}\kappa[P^2 - 2P_m(z)] + O(\kappa^2). \qquad (2)$$

To describe the unloading process, i.e., the process characterized by a pressure decrease with time,

$$\frac{\partial P}{\partial \theta} < 0, \qquad (3)$$

it is necessary to solve Eq. (1) or (2) with an arbitrary function $P_m(z)$. The solution should be sewn together at the point $P = P_m(z)$, $\theta = \theta_m(z)$ with the solution describing the loading wave; then, the functions $P_m(z)$ and $\theta_m(z)$ should be determined.

Nonlinear equations (1) and (2) are difficult to solve. The main difficulty is that the function $P_m(z)$ is unknown and is sought for. Another difficulty is that requirement (3) is preliminarily imposed on the solution for unloading. Below, two asymptotic approaches are used to approximately solve the nonlinear equations.

The first approach (see [5]) only allows the study of the processes for relatively short tracks or small time intervals, but it provides the possibility to overcome many analytical difficulties. The solutions are sought in the form of polynomials in powers of distance or time. Conventional methods of treating asymptotic expansions allow the determination of the limiting distances or intervals within which the aforementioned polynomials can be used for estimating the physical effects.

The second approach is the small-amplitude approximation method (see [6]), which was first proposed by Stokes. The solutions are constructed in the form of expansions in powers of a small parameter proportional to the amplitude. For this approach, the applicability conditions can also be determined.

The first part of the present study is devoted to solving Eqs. (1) and (2) by the two aforementioned methods. It is shown that these methods determine the solution within different intervals of time θ and, thus, complement each other.

The second half of the paper studies the solution of the more general equation

$$(P_z - PP_\theta - \Gamma P_{\theta\theta})_\theta + \kappa(P_z - P_m(z)P_\theta) = 0, \qquad (4)$$

which takes into account the viscosity (absorption) of the medium Γ. This equation is solved by using the exact solution to Eq. (4) at $\kappa = 0$, as well as solutions close to self-similar ones.

Finally, the last section presents the comparison of the hysteretic curves corresponding to different values of the parameters κ and Γ.

SOLUTION IN THE FORM OF A POLYNOMIAL IN θ

Let us begin with the solution for the unloading wave. In the unperturbed case (at $\kappa = 0$), an exact solution is obtained:

$$P = -\frac{\theta}{z-A}, \quad z > A.$$

For this solution, condition (3) is evidently satisfied. According to the aforesaid, we seek the solution to the perturbed equation (2) in the form of an expansion in powers of θ:

$$P = -\frac{\theta}{z-A} + \kappa[\theta P_1(z) + \theta^2 P_2(z)] + O(\kappa\theta^3),$$

where the functions $P_1(z)$ and $P_2(z)$ are to be determined.

The standard procedure of equating the coefficients multiplying different powers of θ leads to the determination of $P_1(z)$ and $P_2(z)$. For the unloading wave, we obtain

$$P = -\frac{\theta}{z-A} - \frac{\kappa\theta}{(z-A)^2}\int P_m(z)(z-A)dz - \frac{\kappa\theta^2}{4}(z-A). \qquad (5)$$

For approximation (5), it is possible to specify the applicability condition. Estimating the subsequent terms of the expansion in powers of θ that did not appear in Eq. (5) and assuming that $A < z < B$, we arrive at the applicability condition

$$\theta < \frac{1}{(B-A)(z-A)^5} \qquad (6)$$

for the explicit formula (5).

For the loading wave, as stated above, the relaxation effect is ignored. Then, the following exact solution is valid:

$$P = \frac{\theta}{B-z}, \quad A < z < B. \qquad (7)$$

We sew together solutions (5) and (7) at the point $P = P_m(z)$, $\theta = \theta_m(z)$. Eliminating the function $P_m(z)$ and using the equality

$$P_m(z)(z-A) = -\theta_m(z) + O(\kappa),$$

which is a consequence of Eq. (5), we arrive at the linear differential equation

$$\theta_m(z) - \frac{4}{z-A}\int\theta_m(z)dz = -\frac{4}{\kappa}\frac{B-A}{(B-z)(z-A)^7}.$$

This equation determines the function $\theta_m(z)$:

$$\theta_m(z) = -\frac{4(B-A)}{\kappa}\frac{1}{(B-z)(z-A)^2} - \frac{16(B-A)}{\kappa}(z-A)^3\int\frac{dz}{(B-z)(z-A)^7}. \qquad (8)$$

Formula (8) and the corresponding formula for $P_m(z)$ are fairly cumbersome. Let us assume that the interval (A, B) under consideration is sufficiently wide and that the z coordinate is close to the left-hand end of this interval:

$$z - A \ll B - A. \qquad (9)$$

Condition (9) allows us to approximately calculate the integral in Eq. (8) and to obtain finite explicit expressions for θ_m and P_m:

$$\kappa\theta_m = B - A\left[1 - \frac{4}{3(z-A)^3(B-A)} + O[(B-A)^{-3}]\right],$$
$$\kappa P_m = 1 - \frac{4}{3(z-A)^3(B-A)} + O[(B-A)^{-3}]. \qquad (10)$$

If we eliminate the quantity $(z–A)$ from the above expressions, we arrive at the boundary curve connecting the points of maximum loads for different values of z. In the leading order, this curve has the form

$$P_m = \frac{1}{B-A}\theta_m + O[(B-A)^{-3}]. \qquad (11)$$

Figure 1 shows the loading and unloading lines for two values of z: $z = z_1$ and $z = z_2$; in addition, Fig. 1 contains the boundary curve given by Eq. (11). According to Eq. (7), when $z_2 > z_1$, the loading line corresponding to $z = z_2$ lies above the loading line corresponding to $z = z_1$. According to Eq. (5), the unloading line corresponding to $z = z_1$ goes down steeper than the unloading line corresponding to $z = z_2$ does.

The formulas derived in this section are valid under conditions (6) and (9).

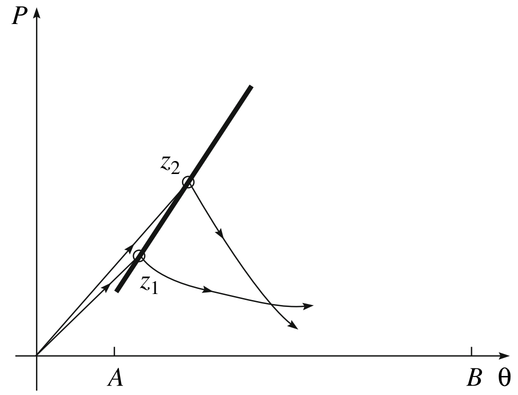

Fig. 1. Loading and unloading lines for $z_1 < z_2$ and relatively small times satisfying condition (6). The thick solid line represents the boundary curve.

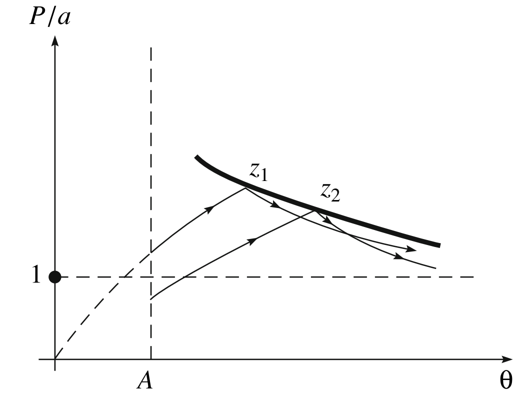

Fig. 2. Loading and unloading lines for $z_1 < z_2$ and relatively large times satisfying conditions (14) and (15). The thick solid line represents the boundary curve.

SMALL-AMPLITUDE APPROXIMATION

Now, we solve Eq. (2) by another method: we use the Stokes small-amplitude ansatz

$$P = au(\theta, z) + a^2 v(\theta, z) + a^3 w(\theta, z) + O(a^4), \quad a \ll 1. \tag{12}$$

Here, u, v, and w are unknown functions. We substitute this expansion into Eq. (2). The resulting leading-order equation shows that the function $u = u(\theta)$ is independent of z.

For the unloading wave, the following conditions should be satisfied:

$$\text{(i) } u(0) = 0 \text{ and (ii) } u'(\theta) < 0.$$

Condition (i) means that P becomes equal to zero simultaneously with θ. Inequality (ii) provides the fulfillment of condition (3). To satisfy these requirements, we set

$$u = \frac{\theta}{\theta - A}, \quad A > 0.$$

Then, we assume that the parameters a and κ are of the same order of smallness and that $P_m = aQ_m$. By determining the subsequent terms of expansion (12), we arrive at the expression

$$P = \frac{a\theta}{\theta - A} - \frac{a^2 A\theta}{(\theta - A)^3}z + a^2 w(\theta, z) + O(a^3). \tag{13}$$

Evidently, the formulas constructed above refer to the case where

$$\theta > A > 0. \tag{14}$$

The condition of applicability of Eq. (13) (analogous to condition (6)) has the form

$$\theta - A > \sqrt{aAz}. \tag{15}$$

For the loading wave, we choose $U = \theta/(\theta + B)$, $B > 0$, which provides the inequality $\partial P/\partial\theta > 0$. Now, calculations yield

$$P = \frac{a\theta}{\theta + B} + \frac{a^2 B\theta z}{(\theta + B)^3} + O(a^3). \tag{16}$$

Sewing together Eqs. (13) and (16) at $P = P_m$ and $\theta = \theta_m$ leads to the elimination of the function $P_m(z)$. Then, we obtain the equation for the difference $\tau = \theta_m(z) - A$:

$$-\frac{A + B}{(A + B + \tau)\tau} + \frac{aBz}{(A + B + \tau)^3} + \frac{aAz}{\tau^3}$$
$$= a^2\left[-\frac{kz(\tau + A)}{2\tau^2} + \frac{k}{\tau}\left(z + A\int\frac{dz}{\tau}\right) + \frac{z^2 A(\tau + 2A)}{\tau^5}\right].$$

This equation contains a small parameter a and a large parameter $A + B$. As a result of asymptotic calculations, we find that, in the leading order, the following relation is valid:

$$\theta_m(z) = A - \frac{az}{2} + O(a^{3/2}). \tag{17}$$

The use of Eqs. (17) and (16) allows us to write down an explicit expression for $P_m(z)$. From Eq. (13), it follows that the boundary curve (analogous to curve (11)) is approximately described by the equation

$$P_m = a\frac{\theta_m}{\theta_m - A}, \quad \theta_m > A. \tag{18}$$

The loading and unloading lines and the boundary curve given by Eq. (18) are shown in Fig. 2.

Let us compare the latter results with the results obtained in the previous section. We assume that, in both cases, the solutions are considered within the same time interval $A < \theta < B$. Solution (5) for the unloading wave corresponds to not-too-large times (see inequality (6) bounding θ from above). Solution (13) obtained for the unloading wave in another way is valid under condition (15), which bounds θ from below. Thus, the two approaches used above yield solutions for different parts of the interval $A < \theta < B$ and, hence, complement each other.

In both Figs. 1 and 2, an increase in z corresponds to an increase in θ_m. This means that, at a greater depth in the medium, time $\theta_m(z)$, at which the maximum pressure $P_m(z)$ is reached, proves to be greater.

INCLUSION OF VISCOSITY

With allowance for the dimensional viscosity Γ, Eq. (1) acquires additional terms:

$$\begin{aligned}&\frac{\partial}{\partial\theta}[P_z - PP_\theta - \Gamma P_{\theta\theta}]\\&+\kappa[P_z - P_m(z)P_\theta - \Gamma P_{\theta\theta}] = 0.\end{aligned} \tag{19}$$

In this section, we assume that

$$\Gamma \ll 1 \tag{20}$$

(i.e., the viscosity is small). When the inequality $\kappa \ll 1$ and inequality (20) are simultaneously satisfied, the last term $\kappa\Gamma P_{\theta\theta}$ in Eq. (19) can be ignored. At the same time, even under condition (20), the term $\Gamma P_{\theta\theta}$ is significant, because it contains the highest-order derivative in Eq. (19). The condition $\kappa \ll 1$ allows us to represent Eq. (19) in the form

$$P_z - PP_\theta - \Gamma P_{\theta\theta} = -\frac{\kappa}{2}[P^2 - 2P_m(z)P] + O(\kappa^2), \tag{21}$$

which is analogous to Eq. (2).

In the leading order (at $\kappa = 0$), we obtain the Burgers equation

$$P_z - PP_\theta - \Gamma P_{\theta\theta} = 0, \tag{22}$$

which has the exact solution

$$P_0 = \frac{1}{z-A}\left[\tanh\frac{\theta}{2\Gamma(z-A)} - \theta\right] \tag{23}$$

at any Γ and A.

When

$$z - A < 1/2\Gamma, \tag{24}$$

the expression for P_0 satisfies the condition

$$\partial P/\partial\theta > 0 \tag{25}$$

and can be used for describing the loading wave. In this case, it is expedient to take into account only two terms of the series expansion:

$$P = \frac{1}{z-A}\left[\left(\frac{1}{2\Gamma(z-A)} - 1\right)\theta - \frac{1}{3}\left(\frac{\theta}{2\Gamma(z-A)}\right)^3\right]. \tag{26}$$

Under the condition

$$z - A > 1/2\Gamma, \tag{27}$$

expression (23) for P_0 satisfies condition (3); i.e., it can describe the unloading wave. However, because Γ is small, condition (27) is rather restrictive. Therefore, it is convenient to construct the solution for unloading in a different way. Equation (22) has a self-similar solution:

$$P = \frac{1}{\sqrt{z-A}}\Phi(y), \quad y = \frac{\theta}{\sqrt{z-A}}, \quad z > A, \tag{28}$$

while equation (21), correspondingly, has a solution close to self-similar one. The substitution of solution (28) leads to the equation

$$\begin{aligned}&\Gamma\Phi'' + \Phi\Phi' + \frac{1}{2}(\Phi + y\Phi')\\&= \kappa\sqrt{z-A}\,\Phi\left(\frac{1}{2}\Phi - \sqrt{z-A}\,P_m\right) + O(\kappa^2).\end{aligned} \tag{29}$$

We construct the solution to Eq. (29) in the form of an expansion in powers of y. In the leading order, for the unloading wave, we obtain

$$\begin{aligned}P = &\frac{1}{\sqrt{z-A}}\left[\left(\frac{\Gamma}{\kappa\sqrt{z-A}}\right)^3 - \frac{1}{2}\frac{\theta}{\sqrt{z-A}}\right]\\&+ O\left(\frac{\kappa\theta}{\sqrt{z-A}}\right).\end{aligned} \tag{30}$$

Sewing together Eqs. (26) and (30) at $\theta = \theta_m$, $P = P_m$, we arrive at the following explicit formulas in the leading order:

$$\theta_m = 2\Gamma\left(\frac{\Gamma}{\kappa}\right)^{1/3}(z-A)^{7/6}\left[1 + \frac{1}{3}\left(\frac{\Gamma}{\kappa}\right)^{2/3}(z-A)^{1/3}\right], \tag{31}$$

$$\begin{aligned}P_m = &\left(\frac{\Gamma}{\kappa}\right)^{1/3}\frac{1}{(z-A)^{2/3}}\\&\times\left[1 - \Gamma(z-A)^{5/6} - \frac{1}{3}\Gamma\left(\frac{\Gamma}{\kappa}\right)^{2/3}(z-A)^{7/6}\right],\end{aligned} \tag{32}$$

which yields the equation for the boundary curve

$$P_m\theta_m^{4/7} = (2\Gamma)^{4/7}\left(\frac{\Gamma}{\kappa}\right)^{11/21}. \tag{33}$$

Curve (33) together with typical loading–unloading lines is shown in Fig. 3.

In connection with Eqs. (26) and (30), it should be noted that, unlike κ and despite condition (20), the viscosity Γ already affects the principal parts of the expressions for P_m and θ_m.

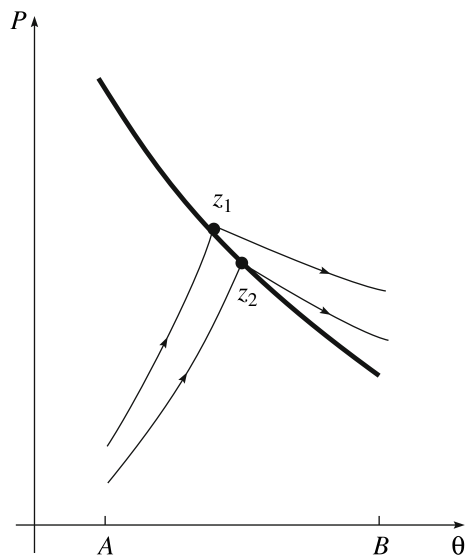

Fig. 3. Loading and unloading lines for $z_1 < z_2$ and a small viscosity. The thick solid line represents the boundary curve.

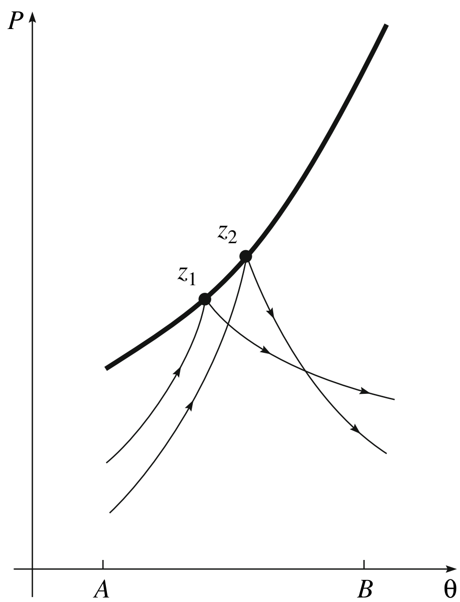

Fig. 4. Loading and unloading lines for $z_1 < z_2$ and a relatively large viscosity. The thick solid line represents the boundary curve.

THE CASE OF A RELATIVELY LARGE VISCOSITY

As before, to describe the unloading wave, we solve Eq. (21). In contrast to the previous section, we assume that the viscosity Γ is relatively large and apply other methods for analyzing this equation. To describe the unloading wave, we use a perturbation of the solution (23) to the Burgers equation. To describe the loading wave, the self-similar solution of type (28) is unsuitable, because, in this case, it is impossible to satisfy condition (25). Therefore, for the loading wave, we use a specific ansatz.

Thus, we seek the unloading wave in the form

$$P = P_0 + \kappa g(\theta, z),$$

where, for P_0, we use representation (26). We assume that Γ is large and that inequality (27) is satisfied. For the function g, on the basis of Eq. (21) we obtain a linear inhomogeneous equation:

$$\Gamma g_{\theta\theta} + (P_0 g)_\theta = \frac{1}{2}(P_0^2 - 2P_m P_0).$$

Solving this equation and performing some calculations, we arrive at the formula

$$P = P_0 + \kappa\left[\frac{\Gamma}{3} + \frac{1}{2}P_m\theta + b\theta^2\right], \quad b \equiv \frac{1}{6(z-A)}\left(1 - \frac{1}{\Gamma(z-A)}\right). \tag{34}$$

Formula (26) for P_0 shows that condition (3) is satisfied in this case.

To determine the loading wave, we apply the ansatz

$$P = \frac{\theta}{B-z} + \frac{c\theta^\alpha}{(B-z)^2}\dots, \quad z > B,$$

where the constants α and c are to be determined. We obtain

$$P = \frac{\theta}{B-z} + \frac{1}{6\Gamma}\frac{\theta^3}{(B-z)^2} + O\left[\frac{\theta^5}{(B-z)^3}\right]. \tag{35}$$

As in Eq. (7), the variable z belongs to the interval (A, B).

By sewing together Eqs. (34) and (35), we arrive at the explicit formulas

$$\theta_m^2 = \frac{2\Gamma(A+B)(B-z)^2}{z-A}, \tag{36}$$

$$P_m^2 = \frac{2\Gamma(A+B)}{z-A}. \tag{37}$$

The boundary curve is determined by the approximate equation

$$P_m = \frac{\theta_m}{B-A} + \frac{3(A+B)}{\kappa\Gamma(B-A)^3}\theta_m^2$$

and is represented in Fig. 4.

Comparing Eqs. (31) and (32) (for small values of Γ) with Eqs. (36) and (37) (for large values of Γ), one can see that, as the viscosity increases, the principal parts of the expressions for P_m and θ_m cease depending on the relaxation coefficient.

DEPENDENCE OF DENSITY ON PRESSURE. HYSTERETIC CURVES

Let us use the dimensionless density and pressure

$$R = \frac{\rho'}{\rho_1}, \quad P = \frac{p'}{p_1}$$

and the dimensionless parameter $\chi = p_0'/\rho_1 c_1^2$. Here, ρ' and p' are the dimensional density and pressure; ρ_1, p_1, and c_1 are the values of density, pressure, and velocity of sound before deformation. The determining equation from [4] (see Eq. (14) in [4]) can be represented in the form

$$R = \chi P - \varepsilon\chi^2 P^2 + \frac{\varepsilon\chi^2}{T_R}, \qquad I = \int_{t_m}^{t} [P - P_m(z)]^2 \exp\left(-\frac{t-t'}{T_R}\right) dt'. \tag{38}$$

Let us consider the case of $\kappa \ll 1$, when the relaxation time T_R is large. The integral I is calculated over a time interval that is small compared to T_R. Therefore, Eq. (38) for the unloading wave takes the form

$$R = \chi P - \varepsilon\chi^2 P^2 + \varepsilon\chi^2\kappa(P_m - P)^2 + O(\varepsilon\kappa^2). \tag{39}$$

At $\kappa = 0$, this formula describes the loading wave.

Formula (39) makes it possible to plot the dependences $R(P)$ for both loading and various unloading processes in an acoustic medium. Let us first consider the influence of the relaxation parameter κ. The calculations carried out above show that the quantity P_m rather weakly depends on κ.

Figure 5 illustrates the loading and various types of unloading processes in the medium under the effect of a pulsed signal. In the absence of relaxation, the loading and unloading processes are represented by curve *1* and straight line segment *2*, respectively. For $0 < \kappa < 1$, the unloading process is described by curve *3*. In this case, the residual strain

$$R_{\text{res}, 0<\kappa<1} = \varepsilon\chi^2\kappa P_m^2 < \varepsilon\chi^2 P_m^2 = R_{\text{res}, \kappa=0} \tag{40}$$

is smaller than that in the absence of relaxation. Note that, in the case of $\kappa > 1$ (small relaxation times), curve *3* will lie above curve *2*.

Proceeding to the inclusion of viscosity Γ, we assume that Eq. (39) remains valid. Then, the maximum pressure decreases:

$$P_{m,\Gamma>0} < P_{m,\Gamma=0}.$$

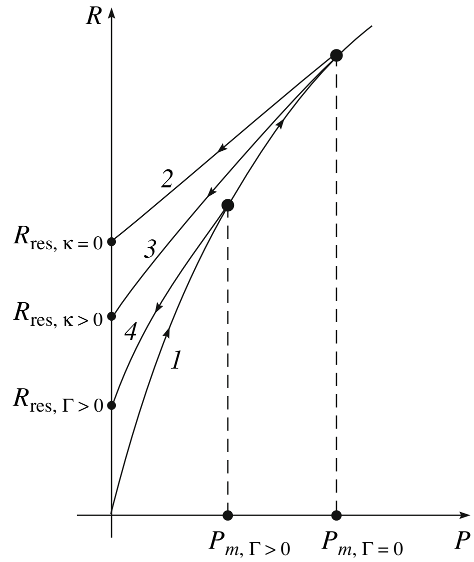

Fig. 5. Dependence of density on pressure for the cases of loading and different types of unloading: (*1*) loading without relaxation, (*2*) unloading without relaxation, (*3*) unloading at $0 < \kappa \ll 1$ (Eqs. (5) and (12)), and (*4*) unloading at $\Gamma > 0$ (Eq. (24)).

Correspondingly, the residual strain also decreases (curve *4* in Fig. 5):

$$R_{\text{res}, \Gamma>0} = \varepsilon\chi^2\kappa P_{m,\Gamma>0}. \tag{41}$$

CONCLUSIONS

Thus, the loading and unloading waves in a relaxing and dissipative nonlinear medium were investigated.

The problem consisted in integrating the evolution equations obtained previously in [4]. The main difficulty in solving the problem was related to its nonlinearity and, what is most important, to the fact that the evolution equations contained the unknown maximum pressure $P_m(z)$, which should be determined as a result of solving the problem. In addition, expressions for the pressure $P(\theta, z)$ should satisfy requirements (25) and (3) for loading and unloading, respectively.

The aforementioned evolution equations were solved using the Stokes small-amplitude approximation, the new asymptotic small-distance (small-time) approach [5], the known exact solution (23) to the Burgers equation, and some special ansatzes, such as expansion (35). For the asymptotic expansions, the conditions of their applicability were determined (see, e.g., inequalities (6) and (15)). The solutions constructed with the use of different schemes were found to belong to different time intervals of the wave process and, hence, to complement each other.

The application of the methods listed above made it possible to obtain explicit approximate solutions for loading and unloading waves in different types of relaxing media with different values of the viscosity (dissipation) parameter. Explicit expressions were obtained for the maximum pressure $P_m(z)$ and the corresponding time $\theta_m(z)$, at which the derivative $\partial P/\partial\theta$ changes sign.

For all of the cases considered above (Figs. 1–4), it was found that, when the depth of penetration into the medium increases, the time $\theta_m(z)$ at which the maximum pressure $P_m(z)$ is reached also increases. The mutual influence of the viscosity and relaxation effects was studied. It was found that, as the viscosity increases, the principal parts of the expressions for P_m and θ_m cease to depend on the relaxation coefficient.

Dependences of density on pressure were determined for different types of media, and the influence of the type of the medium on the shape of the hysteretic curve and on the magnitude of the residual strain were described.

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