Journal of Communications Technology and Electronics, Vol. 47, No. 5, 2002, pp. 501–506. Translated from Radiotekhnika i Elektronika, Vol. 47, No. 5, 2002, pp. 558–563.

ELECTRODYNAMICS AND WAVE PROPAGATION

Analysis of Femtosecond Pulses Based on Equations of the Two-Level Model

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Received March 13, 2001

Abstract—Nonlinear interaction of femtosecond pulses with a medium is considered analytically using the two-level model. Three mutually complementary asymptotic approaches are applied, and numerical calculations are performed. The explicit formulas describing femtosecond-pulse amplitude and phase distortions due to relaxation in the medium are obtained.

INTRODUCTION

Supershort light pulses are of considerable interest for basic research (into ultrafast processes in physics, chemistry, and biology) and applications that involve designing fiber-optic transmission systems. Among the methods of generating supershort light pulses, the nonlinear-optical fiberguide techniques prevail (see, for example, [1, 2]). In the subpicosecond and femtosecond ranges, the nonlinear and dispersion effects should be taken into account much more accurately than in the case of longer pulses. There are certain serious theoretical arguments [3–6] indicating that two additional terms (the nonlinear and dispersion ones) in the nonlinear Schrödinger equation (NSE) enable one to describe the transition to the subpicosecond range. Note that these additional terms in the NSE naturally appear in the asymptotic equation for a pulse envelope [7, 8].

In the femtosecond range, the above improvement of the NSE is not enough, and the interaction of a supershort pulse with a medium should be taken into account. It is conventional to describe this interaction using the two-level model of the medium. In the simplest case, this model leads to the system of equations

$$i\psi_x = \frac{1}{2}\psi_{tt} + n\psi - i\gamma(n\psi)_t + i\beta\psi_{ttt}, \tag{1}$$

$$\sigma n_t + n = |\psi|^2 - r|\psi|^4, \quad \sigma > 0, \quad r > 0, \tag{2}$$

where the NSE is supplemented with a phenomenological equation of relaxation [9, 10]. Here, $\psi(x, t)$ is a complex amplitude of the pulse envelope, $n(x, t)$ is a real function determining the excitation degree of the highest of two quantum levels available. For simplicity, in Eq. (1), we neglect the term describing pulse attenuation during propagation. Being a first-order equation, relaxation equation (2) does not take into account the reverse transition to the nonexcited level. Because of this drawback, the above model can be applied only when the t is not too large.

Of all coefficients β, γ, r, and σ involved in Eqs. (1) and (2), it is most important to study parameter σ, which is related to the relaxation time of a medium in the presence of a supershort pulse field, because the other coefficients have already been investigated, though by other means, in the literature. The present work concentrates mainly on this relaxation term and its influence on the distortion of the pulse shape.

Since equation (2) can be integrated,

$$\begin{aligned} n(x, t) &= \frac{1}{\sigma}\int_{-\infty}^{t} \exp\left[-\frac{1}{\sigma}(t - t')\right] \\ &\times [|\psi(x, t')|^2 - r|\psi(x, t')|^4]dt', \end{aligned} \tag{3}$$

the system of equations under consideration reduces to a single integro-differential equation. However, both the original system of equations and the integro-differential equation obtained are too cumbersome to be analytically investigated in the general case. In what follows, we assume all coefficients β, γ, r, and σ to be small. It can be seen that, in this case, Eqs. (1) and (2) written for the principal-order terms can be replaced with the perturbed NSE,

$$\begin{aligned} S[\psi] &\equiv i\psi_x - \frac{1}{2}\psi_{tt} - |\psi|^2\psi \\ &= \sigma R_1 + \gamma R_2 + \beta R_3 + rR_4, \end{aligned} \tag{4}$$

where

$$R_1 = -(|\psi|^2)_t\psi, \quad R_2 = -i(|\psi|^2\psi)_t,$$
$$R_3 = i\psi_{ttt}, \quad R_4 = -|\psi|^4\psi.$$

Below, our purpose is to analyze the effect of perturbation terms in Eq. (4) (and, especially, the relaxation term σR_1) on the deformation of a solitary pulse. In the subsequent sections, various asymptotic approaches are applied (see [8], [11]). These approaches are complementary and based on the fact that the nonperturbed equation $S[\psi] = 0$ has the two-parameter soliton solution

$$\psi_s = a\frac{\exp\left[-i\frac{b}{2}t + \frac{i}{2}\left(\frac{b^2}{4} - a^2\right)x\right]}{\cosh\left[a\left(t - \frac{1}{2}bx\right)\right]}. \tag{5}$$

Here, parameter a describes the soliton amplitude and inverse width and parameter b describes the soliton velocity (with respect to the general motion at a certain group velocity).

1. PERTURBATION OF THE SOLITON SOLUTION AS A WHOLE

In the principal-order terms, parameters β, γ, r, and σ affect the perturbation of the soliton solution independently. The role of coefficients β and γ has already been clarified [8, 11]. Below, we analyze the influence of relaxation parameter σ.

We represent the phase and amplitude of the anzatz as power series in σ:

$$\psi = a\frac{\exp(i\Phi)}{\cosh Q}, \tag{6}$$

$$\Phi = \Phi_0(x, t) + \sigma\varphi(x, t) + \dots, \tag{7}$$

$$Q = Q_0(x, t) + \sigma q(x, t) + \dots. \tag{8}$$

Functions Φ and Q are real. According to expression (5), we set

$$\Phi_0 = \frac{-b}{2}t + \frac{1}{2}\left(\frac{b^2}{4} - a^2\right)x, \quad Q_0 = at - \frac{1}{2}abx.$$

For correction terms φ and q, we obtain the system of equations

$$\begin{cases} \varphi_{tt} + q_{tt}\tanh Q_0 - b\varphi_t - 2\varphi_x + 2aq_t\left(\dfrac{2}{\cosh^2 Q_0} - 1\right) \\ = 4a^3\dfrac{\sinh Q_0}{\cosh^3 Q_0}, \\ a\varphi_t - \dfrac{b}{2}q_t - q_x + \dfrac{3}{2}abq\dfrac{1}{\sinh Q_0 \cosh Q_0} = 0. \end{cases} \tag{9}$$

The most interesting solution of system (9) corresponds to the case

$$a \gg 1, \tag{10}$$

when the initial amplitude of the soliton is large.

If condition (10) is fulfilled, we should introduce rapidly varying arguments $\tau = at$, $\xi = ax$, and $Q_0 = \tau - \frac{1}{2}b\xi$ into system (9) and set

$$\varphi = a\varphi_0(\xi, \tau) + O(a^0), \quad q = aq_0(\xi, \tau) + O(a^0). \tag{11}$$

This allows us to find that φ_0 depends on $\xi = ax$ only and

$$q_0 = \frac{4}{15}(\cosh^2 Q_0 + 3\ln\cosh Q_0) > 0. \tag{12}$$

An important consequence of the formulas obtained is the explicit description of the soliton amplitude

$$A(Q_0) = \frac{1}{\cosh[Q_0 + a\sigma q_0(Q_0)]}, \quad Q_0 = \tau - \frac{1}{2}b\xi, \tag{13}$$

where $\sigma > 0$ is the dimensionless relaxation time of the medium. Even at small values of $a\sigma$, formula (13) demonstrates a considerable variations of the pulse shape.

For an unperturbed soliton, the amplitude maximum is observed at $Q_0 = 0$ and, at the inflexion point, $\cosh Q_0 = \sqrt{2}$ and $\sinh Q_0 = 1$. It follows from expression (13) that, at $\sigma > 0$, the maximum of $A(Q_0)$ shifts to the left, to the point $Q_0 \approx -0.2a\sigma$. The right inflexion point shifts to the right by $0.1a\sigma$ and downward by $0.4a\sigma$. The left inflexion point shifts considerably to the left by $1.4a\sigma$ and downward again by $0.4a\sigma$. The amplitude curve becomes asymmetric, and its total width decreases.

2. EQUATIONS FOR THE SOLITON PARAMETERS

Let us consider another method for analyzing the influence of perturbation parameters β, γ, r, and σ involved in Eq. (4) on parameters a and b of unperturbed soliton (5). To this end, we write Eq. (4) with a single small parameter, ε:

$$i\psi_x - \frac{1}{2}\psi_{tt} - |\psi|^2\psi = \varepsilon R[\psi], \quad 0 < \varepsilon \ll 1. \tag{14}$$

We assume that, during the signal evolution, formula (5), in the principal-order terms, remains unchanged, with parameters a and b being functions of a slowly varying argument rather than constants as before:

$$X = \varepsilon x. \tag{15}$$

Our next purpose is to derive differential equations for $a(X)$ and $b(X)$.

Let us multiply Eq. (14) by complex-conjugate function ψ^* and subtract from the result the complex-conjugate equation multiplied by ψ. Then, we integrate the obtained equation with respect to t over the infinite interval. Finally, we arrive at

$$i\frac{d}{dx}\int_{-\infty}^{\infty}|\psi|^2 dt = \varepsilon\int_{-\infty}^{\infty}(R\psi^* - R^*\psi)dt. \tag{16}$$

The integral on the left-hand side of Eq. (16) can be calculated with the substitution $\psi = \psi_s$ (see expression (5)):

$$\int_{-\infty}^{\infty}|\psi_s|^2 dt = 2a.$$

As a result, the equation for function $a(X)$ takes the form

$$\frac{da}{dX} = \frac{1}{2i}\int_{-\infty}^{\infty}(R[\psi]\psi^* - R^*[\psi^*]\psi)dt. \tag{17}$$

Next, let us multiply Eq. (14) by ψ_t^* and add the complex-conjugate equation multiplied by ψ_t. The result is integrated again with respect to t over the infinite interval:

$$i\int_{-\infty}^{\infty}(\psi_x\psi_t^* - \psi_x^*\psi_t)dt = \varepsilon\int_{-\infty}^{\infty}(R\psi_t^* + R^*\psi_t)dt.$$

It can be shown that the integral on the left-hand side can be transformed in the following way:

$$\int_{-\infty}^{\infty}(\psi_x\psi_t^* - \psi_x^*\psi_t)dt = \frac{1}{2}\frac{d}{dx}\int_{-\infty}^{\infty}(\psi\psi_t^* + \psi^*\psi_t)dt.$$

Combining the results obtained and substituting expression (5) into them, we find the second differential equation:

$$\frac{d(ab)}{dX} = -\int_{-\infty}^{\infty}(R[\psi]\psi_t^* + R^*[\psi^*]\psi_t)dt. \tag{18}$$

Equations (17) and (18) describe the slow change of soliton parameters in the process of the propagation of a perturbed soliton.

3. PERTURBATION OF THE SOLITON PARAMETERS

Let us analyze, using Eqs. (17) and (18), the dynamics of soliton parameters a and b under the action of the perturbation terms in Eq. (4). Note that the integrals

$$\int_{-\infty}^{\infty}(R\psi^* - R^*\psi)dt \quad\text{and}\quad \int_{-\infty}^{\infty}(R\psi_t^* + R^*\psi_t)dt$$

in formulas (17) and (18) are independent of rapidly varying argument x. Therefore, we can take $x = 0$ in the corresponding integrands.

First, we examine the effect of parameter σ, setting in Eqs. (17) and (18) $\varepsilon = \sigma$, $x_1 = \sigma x$, and $R = R_1 = -\psi(|\psi|^2)_t$. One can see that $\frac{da}{dx_1} = 0$, i.e., the amplitude parameter of the soliton is independent of the coordinate σx. Equation (18) takes the form

$$a\frac{db}{dx_1} = \int_{-\infty}^{\infty}[(|\psi|^2)_t]^2 dt = 4a^6\int_{-\infty}^{\infty}\frac{\sinh^2 at}{\cosh^6 at}dt$$

or

$$\frac{db}{dx_1} = \frac{16}{15}a^4.$$

Thus,

$$b(x_1) = b(0) + \frac{16}{15}a^4\sigma x. \tag{19}$$

It can be seen that the soliton velocity decreases with the growing relaxation time. It is natural that the propagation process slows down due to relaxation.

A variation of parameter b in expression (5) means that, during propagation, the term

$$\begin{aligned}&\frac{16}{15}a^4\sigma x\frac{\partial\psi_s}{\partial b}\\ &= -\frac{16}{15}a^4\sigma x\left\{\frac{i}{2}\left(t - \frac{bx}{2}\right) - \frac{1}{2}ax\tanh\left[a\left(t - \frac{bx}{2}\right)\right]\right\}\psi_s\end{aligned} \tag{20}$$

is added to the standard soliton ψ_s.

As for the influence of small parameters r, β, and γ, we find that, in all three cases, the right-hand sides of Eqs. (17) and (18) are equal to zero. Thus, in the principal-order terms, a variation of parameters r, β, and γ does not change the soliton parameters.

4. THE CAUCHY PROBLEM FOR THE EQUATIONS OF THE TWO-LEVEL MODEL

Let us use one more technique for studying the deformation of supershort soliton-like pulses. This technique (see [11]) allows us to investigate these deformations only on comparatively short propagation distances but, on the other hand, allows us to consider the finite values of perturbation parameters and to overcome many analytical difficulties.

To study the role of relaxation parameter σ, we consider the Cauchy problem for the equations

$$i\psi_x - \frac{1}{2}\psi_{tt} - n\psi = 0, \tag{21}$$

$$\sigma n_t + n = |\psi|^2 \tag{22}$$

with the initial conditions

$$\psi|_{x=0} = a\frac{\exp(-ibt/2)}{\cosh at}, \tag{23}$$

$$n|_{x=0} = \frac{a^2}{\sigma}\int_{-\infty}^{t}\frac{\exp\left[-\frac{1}{\sigma}(t-t')\right]}{\cosh^2 at'}dt' \equiv f(t,\sigma). \tag{24}$$

The conditions (23) and (24) agree with formulas (3) and (5). Since σ is finite, an analysis of the above problem means a deeper insight into the femtosecond range.

Equations (21) and (22) contain the complex-valued function

$$\psi = T(x,t)\exp[iS(x,t)], \quad T > 0, \quad \operatorname{Im} S = 0 \tag{25}$$

and positive function $n(x, t)$. For the quantities T, S, and n, we obtain the system of equations

$$\begin{cases} TS_x + \frac{1}{2}T_{tt} - \frac{1}{2}TS_t^2 + nT = 0, \\ T_x - \frac{1}{2}TS_{tt} - T_tS_t = 0, \\ \sigma n_t + n - T^2 = 0 \end{cases} \tag{26}$$

with the initial conditions

$$T|_{x=0} = \frac{a}{\cosh at}, \quad S|_{x=0} = \frac{-b}{2}t, \quad n|_{x=0} = f(t,\sigma).$$

We search the localized solutions in the form of power series in x:

$$\begin{cases} T = \frac{a}{\cosh at} + xT_1(t) + x^2T_2(t) + \dots, \\ S = \frac{-b}{2}t + xS_1(t) + x^2S_2(t) + \dots, \\ n = f(t,\sigma) + xn_1 + x^2n_2(t) + \dots. \end{cases} \tag{27}$$

The substitution of Eqs. (27) into the principal-order terms of Eqs. (26) allows us to find $T_1(t)$ and $S_1(t)$. We obtain

$$T(x,t) = \frac{a}{\cosh\left[at - \frac{ab}{2}x + O(x^2)\right]}, \tag{28}$$

$$\begin{aligned} S(x,t) &= \frac{-b}{2}t + \left(\frac{b^2}{8} - \frac{a^2}{2}\right)x \\ &+ [f(t,0) - f(t,\sigma)]x + O(x^2). \end{aligned} \tag{29}$$

The agreement of these expressions with formula (5) is evident. In the linear approximation with respect to x, parameter σ does not affect the amplitude but yields the additional term

$$\Delta = [f(t,0) - f(t,\sigma)]x \tag{30}$$

in the phase.

The equations of the next approximation with respect to x allow us to determine functions $S_2(t)$, $T_2(t)$, and $n_1(t)$ in expansions (27). The expression for T_2 contains relaxation parameter σ. After transformations, we find an expression for the amplitude more exact than formula (28):

$$\begin{aligned} T(x,t) &= \frac{a}{\cosh\left(at - \frac{ab}{2}x\right)} \\ &+ x^2a^6\sigma\frac{\sinh at}{\cosh^6 at}(9 - 4\cosh^2 at) + O(x^3). \end{aligned} \tag{31}$$

The second term in this expression is a weak (at small values of $x^2a^6\sigma$) pulse that has a positive polarity only at $0 < at \le 0.97$. Under the influence of this term, which takes into account the relaxation process, the total amplitude, at $x > 0$, decreases and an additional minimum and a weak maximum appear. In principle, the second term in expression (31) corresponds to the second term in formula (20). The exact coincidence is not possible here because of the different asymptotics used in Sections 3 and 4. Expressions (20) and (31) imply the smallness of σ and x, respectively.

Now, let us calculate the distortion of the pulse phase. This distortion is determined by additional term (30) in expression (29). Calculating the difference $f(t, 0) - f(t, \sigma)$ in the principal-order terms, we obtain

$$\Delta = -2\sigma a^3 x\left[\frac{\sinh at}{\cosh^3 at} + \sigma a\left(\frac{2}{\cosh^2 at} - \frac{3}{\cosh^4 at}\right)\right]. \tag{32}$$

Formula (32) quantitatively describes the phase decrease at $t > 0$, and its increase at $t < 0$.

5. NUMERICAL SOLUTION OF THE CAUCHY PROBLEM

In order to quantitatively refine the results presented in Section 4, we numerically solved the Cauchy problem (21)–(24) at $x > 0$. Such numerical solution is important because it does not require any assumptions on the smallness of parameter σ or coordinate x. In calculations, we used an implicit difference scheme (with weighting factors) and the sweep method. In all the cal-

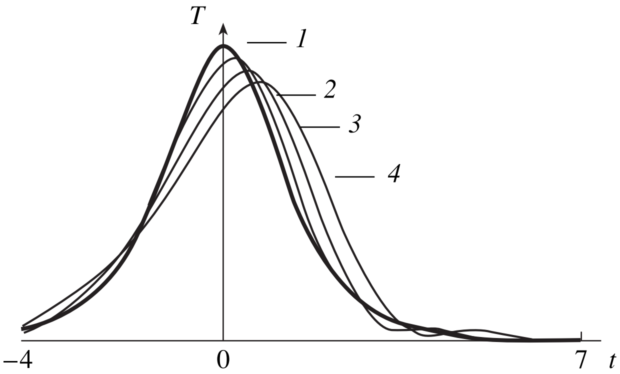

Fig. 1. Pulse amplitude at the fixed value $\sigma = 0.5$ and various x: x = (*1*) 0; (*2*) 2/3; (*3*) 4/3; and (*4*) 2.

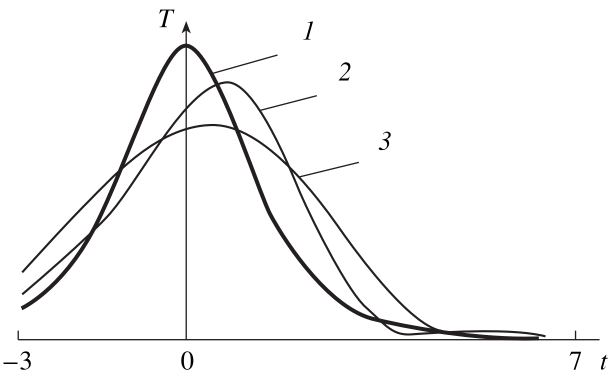

Fig. 2. Pulse amplitude at the fixed value of $x = 2$ and various σ: σ = (*1*) 0; (*2*) 0.5; and (3) 5.

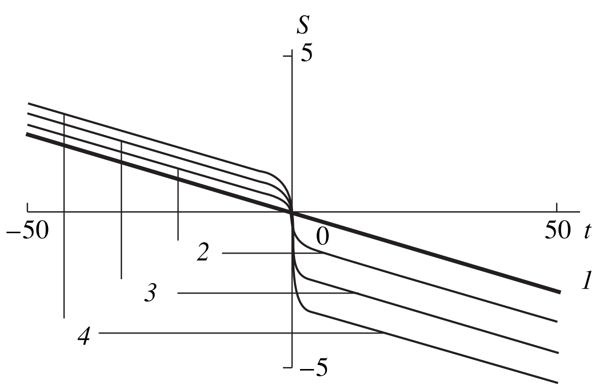

Fig. 3. Pulse phase distortion at the fixed value of $\sigma = 0.5$ and various x: x = (*1*) 0; (*2*) 2/3; (*3*) 4/3; and (*4*) 2.

culations, the starting parameters of the soliton solution (5) were

$$a = 1, \quad b = 0.1. \tag{33}$$

The first group of calculations deals with the description of the pulse amplitude depending on distance x and relaxation parameter σ. Figure 1 demonstrates the curves at the fixed σ equal to 0.5 and three different values of x: 2/3, 4/3, and 2. Figure 2 demonstrates the curves at $x = 2$ and two different values of σ: 0.5 and 5. The initial pulse amplitude is given for comparison in both figures (the heavy line).

The second group of calculations (see Fig. 3) deals with the description of phase distortions of the soliton pulse. In this case, at the fixed value of $\sigma = 0.5$, we considered three different values of x: 2/3, 4/3, and 2. The curves agree with formula (32), which predicts a different polarity of the additional relaxation term in the phase depending on the sign of t, i.e., in the leading and trailing parts of the pulse.

The results of calculation can easily be related to dimensional quantities. The relation between dimensional and dimensionless variables of the NSE is well documented in the literature (see, for example, [3, 9, 10]). Dimensionless σ and x are coupled with dimensional quantities by the formulas [10]

$$\sigma = \tau_1/\tau_0, \quad x = z/z_d, \quad z_d = \tau_0^2|k''|^{-1}. \tag{34}$$

Here, τ_0 is the pulse duration, τ_1 is the characteristic establishment time, z is the actual observation distance, and $|k''|$ is the dispersion factor.

Formulas (34) allow us to indicate the dimensional values corresponding to Figs. 1–3. A realistic value of τ_1 (see [10]) is 6 fs. Then, for example, $\sigma = 0.5$ corresponds to $\tau_0 = 12$ fs. Similarly, at $\tau_0 = 100$ fs and values (33) of soliton parameters, the value $x = 2$ corresponds to the distance $z = 10$ m. At $\tau_0 = 10$ fs, the value $x = 2$ corresponds to $z = 10$ cm.

CONCLUSION

To analyze the behavior of soliton-like pulses in the femtosecond range, three different approaches are applied: (1) perturbation of the soliton solution as a whole; (2) perturbation of soliton parameters; and (3) the analysis of the Cauchy problem with the soliton initial conditions. The general conclusion is that the corresponding results not only agree but supplement each other to a great extent. In view of the special interest attracted by pulse distortion caused by relaxation, the main focus (of all parameters of the system (1), (2)) is on the influence of parameter σ.

The above approaches show that the pulse amplitude decreases due to relaxation. This decrease is described by formulas (13) and (31). The variation of the pulse shape described by formula (13) is considerable. There is a noticeable increase in the pulse width. Formula (31) demonstrates the existence of an additional pulse induced by medium relaxation. The relaxation causes a phase distortion as well. According to formula (32), the leading part of the pulse is slowed down ($\Delta < 0$), while the phase velocity of the trailing part grows. The results obtained using (34) can be related to the actual dimensional quantities.

ACKNOWLEDGMENTS

The authors are grateful to S.A. Vakulenko and V.A. Eremenko for valuable discussions.

This work was supported by the Russian Foundation for Basic Research, project no. 02-02-16046.

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