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Dedicated to the memory of Vladimir Borovikov

Fine Structure of the Soliton Signal in the Magnetic Chain in an External Magnetic Field near a Critical Value

I. A. Molotkov

Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation of Russian Academy of Sciences, Troitsk, Moscow Region, 142190 Russia

E-mail: imolotkov@mtu-net.ru

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Abstract. As is known, there is a critical magnetic field which separates two principally different zones for the soliton signal propagation in magnetic chains, the sine–Gordon zone and the Heisenberg zone. We investigate the fine structure of these signals in a neighborhood of the critical field with nonzero soliton velocity. Explicit formulas both for the azimuthal kink and meridional soliton are obtained. These formulas take into account the nonlinear interaction of soliton structures.

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1. INTRODUCTION

The model in question is a classical (easy plane) isotropic magnetic chain in an external magnetic field near a critical value. The aim of the present paper is to complete the theoretical investigation of this model. Earlier investigations were published in [1–3]. The problem is of actual importance because it is related to inelastic neutron scattering experiments [4, 5] on the planar ferromagnet $CsNiF_3$.

The problem treated here corresponds to the study of the dimensionless Hamiltonian equations of motion,

$$\theta_t = \varphi_{zz}\cos\theta - 2\theta_z\varphi_z\sin\theta - \sin\varphi, \tag{1a}$$

$$\varphi_t = -\theta_{zz}(\cos\theta)^{-1} - \varphi_z^2\sin\theta + \lambda\sin\theta + \tan\theta\cos\varphi \tag{1b}$$

under the boundary conditions $\varphi_{\zeta\to-\infty} \to 0$, $\varphi_{\zeta\to+\infty} \to 2\pi$, and $\theta_{\zeta\to\pm\infty} \to 0$, where $\zeta = z - ut$. Here t is time and z is the coordinate along the chain. Equations (1) describe a nonlinear interaction between an azimuthal kink φ and a meridional soliton θ in the presence of an external magnetic field characterized by the dimensionless parameter λ^{-1}. The desired functions φ and θ are the spherical components for the spin vector. For details concerning system (1), see, e.g., Section 4 of [1].

The following discussion is not limited to zero soliton velocity u (as was the case in earlier works). This is ensured by introducing the independent variable $\zeta = z - ut$. As is known (see [1, 2]), the full spectrum of soliton-like excitations in a realistic magnetic chain has three branches in the low magnetic field regime (sine–Gordon solitons) and only one branch at high magnetic fields beyond a critical field (the Heisenberg soliton). In the dimensionless notation we use, the critical field in question has the value $\lambda = 3$ in the simplest case $u = 0$. We denote this critical value by $3 + \mu(u)$ for $u \neq 0$.

Our principal goal is to study and describe the fine structure of the soliton signal, i.e., to seek a solution of equation (1) for fields close to the critical value and for small soliton velocity $u \neq 0$. We have two small independent parameters Λ and u in our asymptotic investigation. Here Λ is the deviation of the magnetic field in comparison with its critical value. Namely, assume that

$$1/\lambda = 1/(3 + \mu(u) + \Lambda). \tag{2}$$

Here we always have $u > 0$. The value $\Lambda > 0$ corresponds to a decrease of the external field as compared to the critical magnitude, whereas the value $\Lambda < 0$ corresponds to an increasing external field. We have the range of sine–Gordon solitons for $\Lambda+\mu>0$ and the range of Heisenberg solitons for $\Lambda+\mu<0$.

A significant novelty of the present paper as compared with the previous publications is that the nonzero parameters u and Λ are simultaneously taken into account. Another important novelty is in a consistent treatment of the nonlinear interaction between the azimuthal kink (AK) and the meridional soliton (MS). We also widely use asymptotic approaches (see, e.g., [8, 9]).

2. TRANSFORMATION OF THE MOTION EQUATIONS

In the motion equations (1), we pass to the independent variable $\zeta = z - ut$ and substitute expression (2) for λ^{-1} into (1). We also separate the stationary part $\Phi(\zeta)$ of the solution of the motion equations and assume that

$$\varphi = \Phi(\zeta) + \alpha(\zeta), \qquad \theta = \beta(\zeta). \tag{3}$$

Here

$$\sin\frac{\Phi}{2} = \frac{1}{\cosh\zeta}, \quad \Phi_{\zeta\to-\infty}\to 0, \quad \Phi_{\zeta\to+\infty}\to 2\pi. \tag{4}$$

New equations are of the form

$$L_1\alpha = F_1(\alpha,\beta,\zeta,u), \tag{5a}$$

$$L_2\beta = F_2(\alpha,\beta,\zeta,u,\Lambda), \tag{5b}$$

and they contain linear operators $L_1 = d^2/d\zeta^2 - \big(1-2\cosh^{-2}\zeta\big)$ and $L_2 = d^2/d\zeta^2 - \big(4-6\cosh^{-2}\zeta\big)$. The right-hand sides F_1 and F_2 represent expansions in powers of α, β, and their derivatives,

$$\begin{aligned} F_1 &= -u\beta' - \frac{\sinh\zeta}{\cosh^2\zeta}\beta^2 + \frac{4}{\cosh\zeta}\beta\beta' + \frac{1}{12}\frac{\sinh\zeta}{\cosh^2\zeta}\beta^4 - \frac{2}{3\cosh\zeta}\beta^3\beta' + 2\alpha'\beta\beta' \\ &\quad + \frac{\sinh}{\cosh^2\zeta}\alpha^2 + \frac{1}{2}\alpha''\beta^2 + O\big(\beta^5,\alpha\beta^3,\alpha^{3/2}\big), \\ F_2 &= \frac{2u}{\cosh\zeta} + \Lambda\beta - \frac{4}{\cosh\zeta}\alpha'\beta - \frac{1}{6}\Big(13 - \frac{18}{\cosh^2\zeta}\Big)\beta^3 + \frac{2\sinh\zeta}{\cosh^2\zeta}\alpha\beta - u\Big(\frac{1}{\cosh\zeta}\beta^2 - \alpha'\Big) \\ &\quad - \frac{2}{3}\Lambda\beta^3 + O\left(\alpha\beta^3,\beta^5,\alpha^2\beta,u\beta^4,u\alpha\beta\right). \end{aligned}$$

Here and below, we use notation of type $O(x,y,z) = O(x) + O(y) + O(z)$.

In contrast to the corresponding equations in [1, 3], the system of equations (15) takes into account nonlinear interactions between AK and MS, even at a nonzero soliton velocity.

The operators L_1 and L_2 have the eigenfunctions $A\cosh^{-1}\zeta$ and $B\cosh^{-2}\zeta$, respectively, which vanish as $\zeta\to\pm\infty$. The constants A and B characterize the excitation degrees of AK and MS. Therefore, the second step of the transformation of the motion equations is reduced to the relations

$$\alpha = A\cosh^{-1}\zeta + v(\zeta), \tag{6a}$$

$$\beta = B\cosh^{-2}\zeta + w(\zeta), \tag{6b}$$

where v and w are new unknown functions. This leads to the new equations

$$L_1 v = \tilde F_1(v,w,A,B,u,\Lambda), \tag{7a}$$

$$L_2 w = \tilde F_2(v,w,A,B,u,\Lambda), \tag{7b}$$

where the operators $L_{1,2}$ have the same form as in (5) above. The right-hand sides $\tilde F_{1,2}$ are very cumbersome. The functions v and w must vanish for $\zeta\to\pm\infty$.

In formulas (6), not only the correction terms $v(\zeta)$ and $w(\zeta)$ but also the coefficients A and B of the main terms are unknown. Therefore, it is necessary to determine these coefficients by specific conditions, i.e., orthogonality conditions between the right-hand sides $\tilde{F}_{1,2}$ and the eigenfunctions of the operators $L_{1,2}$. These orthogonality conditions correspond to relations of the following type:

$$f(A, B, u, \Lambda) = 0, \qquad g(A, B, u, \Lambda) = 0. \tag{8}$$

The system of equations (8) describes the very dependence of A and B on the parameters u and Λ. However, this dependence is very complicated in general. We therefore consider a particular situation defined by an external field close to its critical value and a small soliton velocity. In this case, system (8) is more transparent.

3. DEPENDENCE OF THE CRITICAL MAGNETIC FIELD ON THE SOLITON VELOCITY. DIFFERENT BRANCHES OF SOLUTION

Equations (8) are uniquely solvable with respect to A and B as functions of u and Λ if the Jacobian

$$\Delta \equiv \begin{vmatrix} f_A & f_B \\ g_A & g_B \end{vmatrix} \tag{9}$$

is nonzero. For

$$\Delta(u, \Lambda) = 0, \tag{10}$$

the unique solvability fails to hold. Note that (10) is a bifurcation equation, and it defines the branching line on the plane (u, Λ). In other words, this equation gives the desired dependence of the critical magnetic field $(3 + \mu(u))^{-1}$ on the soliton velocity u, i.e., the function $\mu(u)$.

Let us consider the dependence of (10) for small u and Λ. It can readily be seen that the coefficients A and B in (6) are small as well. However, exact relations for the size of A, B, u, and Λ near the branching line are not established yet.

The orthogonality conditions (8) become

$$f = c_1 uB + c_2 \Lambda B^2 + c_3 B^4 + c_4 AB^2 + O\left(uB^3, \Lambda B^3, u^2, u\Lambda B\right) = 0, \tag{11}$$

$$g = d_1 u + d_2 \Lambda B + d_3 B^3 + O\left(uB^2, \Lambda B^2\right) = 0. \tag{12}$$

Here c_i and d_i are well-defined numerical coefficients, in particular, $d_1 = \pi$, $d_2 = \pi/2$, and $d_3 = -4/7$. The leading-order equation (10) becomes

$$g_B = d_2 \Lambda + 3 d_3 B^2 = 0. \tag{13}$$

To find the branching line, the coefficient B can be taken from the system (12) and (13). This, together with above value of d_i, leads to the following new result:

$$\mu(u) = 6(7\pi)^{-1/3} u^{2/3} + O\big(u^{4/3}\big) \cong 2.12\, u^{2/3}. \tag{14}$$

Since $\mu(u)$ is positive, the critical field decreases as the soliton velocity increases.

For $\Lambda + \mu(u) > 0$, in connection with formula (2) (see also Fig. 1), we are in the sine–Gordon regime and have three different solutions for the coefficient B, which determines the amplitude of the MS, namely,

$$B_1 = -\frac{2u}{\Lambda + \mu(u)} + O\Big(\frac{u^2}{(\Lambda + \mu)^{5/2}}\Big), \tag{15}$$

$$B_{2,3} = \pm\sqrt{\frac{7\pi}{8}\,(\Lambda + \mu(u))} + \frac{u}{\Lambda + \mu(u)} \cong \pm 1.66\sqrt{\Lambda + \mu(u)} + \frac{u}{\Lambda + \mu(u)}. \tag{16}$$

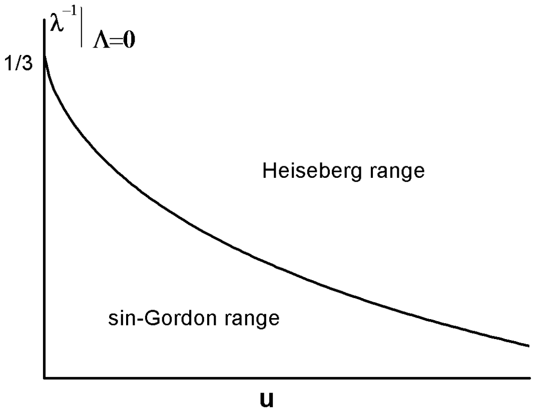

Fig. 1. Dependence on the soliton velocity of the field critical value.

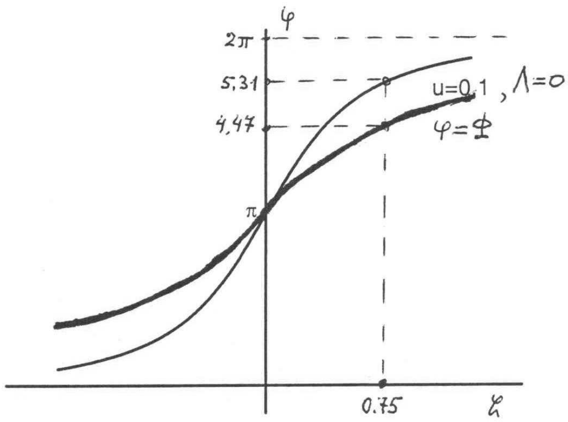

Fig. 2. Typical deformation of AK (3rd branch, $\Lambda = 0$, $u = 0.1$).

The appearance of three branches of the solution in the sine–Gordon range was predicted in [2]. However, the approaches used in [2] and here are quite different. C. Etrich and H.-J. Mikeska made their conclusions about branching on the basis of an expression for the spin correlation function. We obtain explicit formulas (14)–(16) by means of the direct solution of the Hamiltonian equations (1).

For $\Lambda + \mu < 0$, we deal with the Heisenberg regime (Fig. 1). In this regime only one solution B_1 remains.

Also equations (11) and (12) permit to establish relationship between the orders of the desired values A and B. It follows from equation (11) that the coefficient A in (6a) is of the order of B^2 for the small u and Λ. Therefore, we can now suppose in the final expressions (3) that the stationary kink $\Phi(\zeta)$ is the largest, the term $\beta(\zeta)$ is the next in magnitude, and the smallest function $\alpha(\zeta)$ has the order of $\beta^2(\zeta)$.

4. BEHAVIOR OF AK NEAR THE CRITICAL VALUE OF THE EXTERNAL FIELD

Equation (5a), under the additional condition that $\alpha_{\zeta\to\pm\infty} \to 0$, has the following solution (in the leading order):

$$\alpha(\zeta) = \frac{B^2}{2}\frac{1+2\cosh^2\zeta}{\cosh^4\zeta}\sinh\zeta - \frac{2}{3}uB\frac{\arctan(\sinh\zeta)}{\cosh\zeta}.$$

We must substitute $\alpha(\zeta)$ into (3), where it gives a correction to the stationary solution $\Phi(\zeta)$ for $u = 0$. Then the final definitive formula for AK is

$$\varphi = \Phi(\zeta) + \frac{B^2}{2}\frac{1+2\cosh^2\zeta}{\cosh^4\zeta}\sinh\zeta - \frac{2}{3}uB\frac{\arctan(\sinh\zeta)}{\cosh\zeta}. \tag{17}$$

We must substitute here the values of B given by expressions (15) and (16). Then we obtain three branches of AK in the sine–Gordon range for $\Lambda + \mu > 0$ and a unique branch $B = B_H$ in the Heisenberg range for $\Lambda + \mu < 0$.

Formula (17) for AK takes into account the nonlinear action of MS with amplitude B and the dependence of external field on the velocity u and on the deviation Λ in comparison with the critical value of the field. In [3], the second term of the right-hand side of (17) was already derived; however, the coefficient B remained unknown at the time. The third term in this formula, which is very important, is completely new.

The stationary kink (boldface line on Fig. 2) is deformed because of the nonlinear interactions. The typical form of this deformation can also be seen on Fig. 2 (thin line), for example, on the third branch for $\Lambda = 0$ and $u = 0.1$. The nonlinear deformation of the stationary kink is maximal for $\zeta = 0.75$. Figures 3 and 4 give an idea of the fine structure of the soliton signal. Here we see

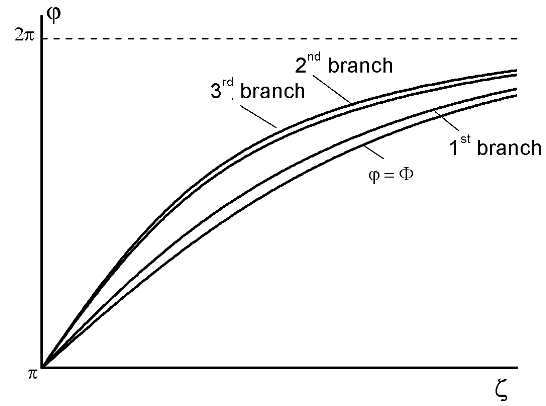

Fig. 3. Essential parts of AK graphs for $\Lambda = 0$ and $u = 0.1$.

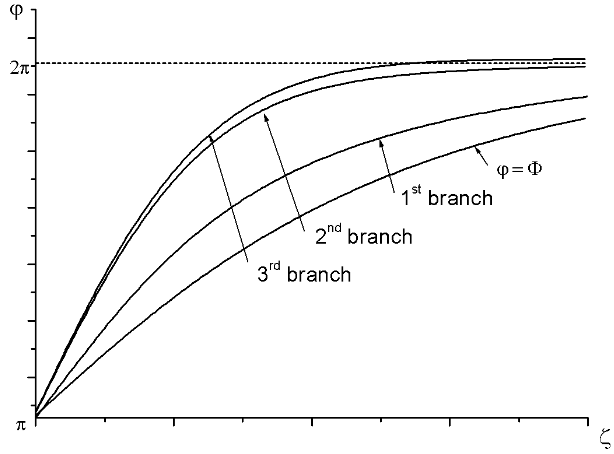

Fig. 4. Essential parts of AK graphs for $u = 0.1$ and $\Lambda = 0.1$.

essential parts of AK graphs for different possible branches of solutions and for different values of the small parameters u and Λ.

The addition of $\alpha(\zeta)$ to the stationary solution can change the soliton velocity u in dependence on the strength of the nonlinearity of the self-action. However, in the case treated here, this effect is absent as long as $\alpha(0) = 0$.

The change of the AK-shape is characterized by the derivative

$$\left.\frac{d\varphi}{d\zeta}\right|_{\zeta=0} = 2 + \frac{3}{2}B^2 - \frac{2}{3}uB \equiv l^{-1}. \tag{18}$$

This derivative describes the velocity of the transition from one spin state to another. It therefore defines a dimensionless thickness l of the domain wall as well. If we use (15) and (16), then formula (18) yields a dependence of this thickness on the values of u and Λ. The analysis of such a dependence shows that the nonlinear interaction with MS leads to a reduction of the thickness l of the domain wall for the AK. This effect is appreciable especially in the sine–Gordon regime, and all the more, notable for the second and third branches of the solutions. The decrease of the thickness grows appreciably as Λ increases, whereas its dependence on the parameter u is much weaker.

5. BEHAVIOR OF MS NEAR THE CRITICAL VALUE OF THE EXTERNAL FIELD. PROPERTIES OF BRANCHES OF THE SOLUTION

The properties of MS are defined by formula (6b). We must substitute expressions (15) and (16) into the main term $B\cosh^{-2}\zeta$ of (6b).

The behavior of the three different branches of the solutions for MS in the sine–Gordon range ($\Lambda + \mu > 0$) is shown in Figs. 5 and 6. The coefficient B_1 determines the behavior of the first branch. This coefficient is negative, its absolute value is comparatively small, and it increases as Λ increases and decreases as u increases. The second and the third branches are related to the coefficients B_2 and B_3, whose signs are opposite to that of the leading order. The absolute values of the amplitudes of the branches have higher order of magnitude than the corresponding values for the first branch.

We can describe the initial behavior of the amplitudes. For $u = 0$, we have $B_1 = 0$ and $B_{2,3} = \pm 1.66\Lambda^{1/2}$. For $\Lambda = 0$, we have $B_1 = -0.95u^{1/3}$ and $B_{2,3} = \pm 2.48u^{1/3}$. The corresponding graphs are shown on Figs. 5 and 6.

In the Heisenberg range ($\Lambda + \mu < 0$), there is a unique first branch only. In contrast to the sine–Gordon range, the coefficient B_1 is positive here.

The branching of solutions and the formation of fine structures of soliton signal are manifested maximally when considering the MS. In the case of AK, these effects are masked by the stationary part $\Phi(\zeta)$ of the solution.

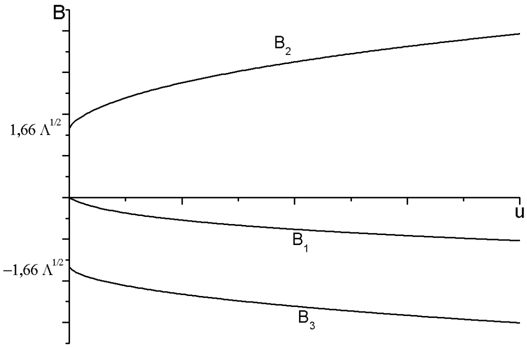

Fig. 5. Dependence of the MS amplitude on u for a fixed $\Lambda > 0$.

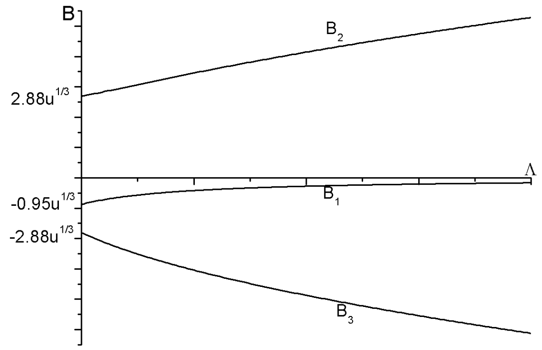

Fig. 6. Dependence of the MS amplitude on Λ for a fixed u.

6. CONCLUSIONS

The above formulas (3), (6), and (15)–(17) give an exhaustive description of the behavior of all the branches of AK and MS for not too large deviations Λ of the external fields from their critical value and for not too large soliton velocity u. Formulas (6) and (15)–(17) are new and give the dependence of AK and MS on Λ and u for the first time.

Formula (18) characterizes the change of the kink shape and of the velocity of transition from one spin state to another and the corresponding decrease of the thickness of the domain of the wall.

Formula (14) describing the dependence of the critical magnetic field on the soliton velocity has been derived here for the first time.

It is possible to use the above approach to generalize and modify the problem in question. For example, similar soliton signals in easy-plane systems with small additional anisotropy can be studied. It is also possible to estimate the energy for the processes under consideration.

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